

COBORDISM INVARIANCE OF THE INDEX OF CALLIAS-TYPE OPERATORS

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ABSTRACT. We introduce a notion of cobordism of Callias-type operators over complete Riemannian manifolds and prove that the index is preserved by such a cobordism. As an application we prove a gluing formula for Callias-type index. In particular, a usual index of an elliptic operator on a compact manifold can be computed as a sum of indexes of Callias-type operators on two non-compact, but topologically simpler manifolds. As another application we give a new proof of the relative index theorem for Callias-type operators, which also leads to a new proof of the Callias index theorem.

1. INTRODUCTION

The study of the index of a Dirac-type operator with potential $B = D + \Phi$ on a complete Riemannian manifold M was initiated by Callias, [10], and further studied by many authors, cf, for example, [3], [8], [2], [9]. A celebrated Callias-type index theorem discovered by these authors in different forms states that the index of a Callias-type operator can be computed as an index of a certain operator induced by it on a compact hypersurface. Recently the interest in Callias-type index theory was revived partially in a relation to the study of the moduli space of monopoles over non-compact manifolds. Several generalizations and new applications of the Callias-type index theorem were obtained, [16], [11], [21], [17].

In this paper we define a class of cobordisms between Callias-type operators and show that the Callias-type index is preserved by this class of cobordisms. The proof of this theorem is similar to the proof of the cobordism invariance of the index on a compact manifold, given in [5], but a more careful analysis is needed. We also present several applications of this result.

Suppose $\Sigma \subset M$ is a compact hypersurface, such that $M \setminus \Sigma$ is a disjoint union of two submanifolds M_1 and M_2 . We assume that Σ lies outside of the *essential support* of the potential Φ . Roughly that means that the restriction of Φ^2 to Σ is strictly positive, cf. Definition 2.2 for a more precise definition. We endow M_1 and M_2 with complete Riemannian metrics and denote by B_1 and B_2 the operators induced by B on M_1 and M_2 respectively. The gluing formula states that

$$\text{ind}(B) = \text{ind}(B_1) + \text{ind}(B_2).$$

This formula should be compared with a similar gluing formula for equivariant Dirac operator with potential obtained in [4]. We remark that the gluing formula is useful and non-trivial even in the case when the original manifold M is compact. The proof of the gluing formula is obtained by constructing a cobordism between M and $M_1 \sqcup M_2$.

As a second application of the cobordism invariance of the Callias index, we prove a version of the relative index theorem for Callias-type operator. The relative index theorem was first proven by Gromov and Lawson [13]. Since then many different reformulations and many different proofs were suggested by different authors. In this paper we prove a version of the relative index theorem suggested by [9]. Notice, that Anghel [2] used the relative index theorem to prove a Callias-type index theorem. Hence, our new proof of the relative index theorem leads to a new proof of the Callias-index theorem, based on the cobordism invariance of the Callias-index.

[†]Supported in part by the NSF grant DMS-1005888.

2. THE MAIN RESULTS

2.1. Callias-type operators. Let (M, g) be a complete Riemannian manifold (possibly with boundary) with metric g . M is endowed with a Hermitian vector bundle E . We denote by $C_0^\infty(M, E)$ the space of smooth sections of E with compact support, and by $L^2(M, E)$ the Hilbert space of square-integrable sections of E which is the completion of $C_0^\infty(M, E)$ with respect to the norm $\|\cdot\|$ induced by the L^2 -inner product

$$(s_1, s_2) = \int_M \langle s_1(x), s_2(x) \rangle_{E_x} d\text{vol}(x), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_{E_x}$ denotes the fiberwise inner product and $d\text{vol}(x)$ is the canonical volume form induced by the metric g .

Let $D : C_0^\infty(M, E) \rightarrow C_0^\infty(M, E)$ be a first-order formally self-adjoint elliptic differential operator and let $\Phi \in \text{End}(E)$ be a self-adjoint bundle map. Then $D + \Phi$ is a first-order elliptic operator, and

$$(D + \Phi)^2 = D^2 + \Phi^2 + [D, \Phi]_+, \quad (2.2)$$

where

$$[D, \Phi]_+ := D\Phi + \Phi D$$

is the anti-commutator of the operators D and Φ .

Definition 2.2. We say that $D + \Phi$ is a (generalized) *Callias-type operator* if

- (i) $[D, \Phi]_+$ is a zeroth order differential operator, i.e. a bundle map;
- (ii) there is a compact subset $K \Subset M$ and a constant $c > 0$ such that

$$|(\Phi^2 + [D, \Phi]_+)(x)| \geq c$$

for all $x \in M \setminus K$. Here $|(\Phi^2 + [D, \Phi]_+)(x)|$ denotes the operator norm of the linear map $(\Phi^2 + [D, \Phi]_+)(x) : E_x \rightarrow E_x$. In this case, the compact set K is called the *essential support* of $D + \Phi$.

2.3. The index of a \mathbb{Z}_2 -graded Callias-type operator. Suppose that $E = E^+ \oplus E^-$ is a \mathbb{Z}_2 -graded vector bundle and that D and Φ are odd with respect to this grading. This means that with respect to this decomposition we have

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & \Phi^- \\ \Phi^+ & 0 \end{bmatrix}.$$

From now on, we suppose that M has no boundary and D satisfies the following assumption.

Assumption 2.4. There exists a constant $k > 0$ such that

$$0 < |\sigma(D)(x, \xi)| \leq k\|\xi\|, \quad \text{for all } x \in M, \xi \in T_x^*M \setminus \{0\}, \quad (2.3)$$

where $\|\xi\|$ denotes the length of ξ defined by the metric g , $\sigma(D)(x, \xi) : E_x^\pm \rightarrow E_x^\mp$ is the leading symbol of D .

An interesting class of examples of operators satisfying (2.3) is given by Dirac-type operators.

By [13, Theorem 1.17], (2.3) implies that D and $D + \Phi$ are essentially self-adjoint operators with initial domain $C_0^\infty(M, E)$. We view $D + \Phi$ as an unbounded operator on $L^2(M, E)$. By a slight abuse of notation we also denote by $D + \Phi$ the closure of $D + \Phi$. Let $\|\cdot\|$ denote the norm on $L^2(M, E)$ induced by (2.1).

Remark 2.5. It's easy to see from (2.2) that a Callias-type operator $D + \Phi$ satisfying Assumption 2.4 is *invertible at infinity*, i.e.,

$$\|(D + \Phi)s\| \geq \sqrt{c}\|s\|, \quad \text{for all } s \in L^2(M, E), \text{ supp}(s) \cap K = \emptyset. \quad (2.4)$$

It follows from [1, Theorem 2.1] and Remark 2.5 that

Lemma 2.6. *If Assumption 2.4 is satisfied, then a Callias-type operator $D + \Phi$ is Fredholm.*

Thus $\ker(D + \Phi) = \ker(D^+ + \Phi^+) \oplus \ker(D^- + \Phi^-) \subset L^2(M, E)$ is finite-dimensional, and the index,

$$\text{ind}(D + \Phi) := \dim \ker(D^+ + \Phi^+) - \dim \ker(D^- + \Phi^-) \quad (2.5)$$

is well-defined.

2.7. Cobordism of Callias-type operators. We now introduce a class of non-compact cobordisms similar to those considered in [4, 7, 12, 14]. One of the main results of this paper is that the index (2.5) is preserved by this class of cobordisms.

Definition 2.8. Suppose $(M_1, E_1, D_1 + \Phi_1)$ and $(M_2, E_2, D_2 + \Phi_2)$ are two triples which are described as above. $(W, F, \tilde{D} + \tilde{\Phi})$ is a *cobordism* between them if

- (i) W is a complete manifold with boundary ∂W and there is an open neighborhood U of ∂W and a metric-preserving diffeomorphism

$$\phi : (M_1 \times (-\epsilon, 0]) \sqcup (M_2 \times [0, \epsilon)) \rightarrow U. \quad (2.6)$$

In particular, ∂W is diffeomorphic to the disjoint union $M_1 \sqcup M_2$.

- (ii) F is a vector bundle (may not be graded) over W , whose restriction to U is isomorphic to the lift of E_1 and E_2 over $(M_1 \times (-\epsilon, 0]) \sqcup (M_2 \times [0, \epsilon))$;
- (iii) $\tilde{D} + \tilde{\Phi} : C_0^\infty(W, F) \rightarrow C_0^\infty(W, F)$ is a Callias-type operator with \tilde{D} satisfying Assumption 2.4, and takes the form

$$\tilde{D} + \tilde{\Phi} = D_i + \gamma \partial_t + \Phi_i \quad (2.7)$$

on U , where t is the normal coordinate and $\gamma|_{E_i^\pm} = \pm\sqrt{-1}$, $i = 1, 2$.

If there exists a cobordism between $(M_1, E_1, D_1 + \Phi_1)$ and $(M_2, E_2, D_2 + \Phi_2)$ then the operators $D_1 + \Phi_1$ and $D_2 + \Phi_2$ are called *cobordant*.

Remark 2.9. If M_2 is the empty manifold, then $(W, F, \tilde{D} + \tilde{\Phi})$ is a *null-cobordism* of $(M_1, E_1, D_1 + \Phi_1)$. In this case the operator $D_1 + \Phi_1$ is called *null-cobordant*.

Remark 2.10. Let E_2^{op} denote the vector bundle E_2 with opposite grading, namely $E_2^{\text{op}\pm} = E_2^\mp$. Consider the vector bundle E over $M = M_1 \sqcup M_2$ induced by E_1 and E_2^{op} . Let $D + \Phi : C_0^\infty(M, E^\pm) \rightarrow C_0^\infty(M, E^\mp)$ be the operator such that $D|_{M_i} = D_i$, $\Phi|_{M_i} = \Phi_i$, $i = 1, 2$. Then $(W, F, \tilde{D} + \tilde{\Phi})$ makes $D + \Phi$ null-cobordant.

2.11. Cobordism invariance of the index. We now formulate the main result of the paper:

Theorem 2.12. *Let $D_1 + \Phi_1$ and $D_2 + \Phi_2$ be cobordant Callias-type operators. Then*

$$\text{ind}(D_1 + \Phi_1) = \text{ind}(D_2 + \Phi_2).$$

By Remark 2.10, this is equivalent to the following

Theorem 2.13. *The index of a null-cobordant Callias-type operator $D + \Phi$ is equal to zero.*

2.14. An outline of the proof of Theorem 2.13. Sections 3-5 deal with the proof of Theorem 2.13. We use the method of [5], [6] with necessary modifications.

Suppose $(W, F, \tilde{D} + \tilde{\Phi})$ is a null-cobordism of $(M, E, D + \Phi)$. In Section 3, we denote by \tilde{W} the manifold obtained from W by attaching a semi-infinite cylinder $M \times [0, \infty)$. Then for small enough number $\delta > 0$ we construct a family of Fredholm operators $\mathbf{B}_{a,\delta}$ on \tilde{W} whose index is independent of $a \in \mathbb{R}$.

An easy computation, cf. Lemma 3.7, shows that for $a \ll 0$ the operator $\mathbf{B}_{a,\delta}^2 > 0$. Hence, its index is equal to 0. Hence,

$$\text{ind } \mathbf{B}_{a,\delta} = 0, \quad \text{for all } a \in \mathbb{R}. \quad (2.8)$$

In Sections 4 and 5 we study the operator $\mathbf{B}_{a,\delta}$ for $a \gg 0$. It turns out that the sections in the kernel of this operator are concentrated on the cylinder $M \times [0, \infty)$ near the hypersurface $M \times \{a\}$. Then we construct an operator $\mathbf{B}_\delta^{\text{mod}}$ on the cylinder $M \times \mathbb{R}$, whose restriction to a neighborhood of $M \times \{0\}$ is very close to the restriction of $\mathbf{B}_{a,\delta}$ to a neighborhood of $M \times \{a\}$. In a certain sense, $\mathbf{B}_\delta^{\text{mod}}$ is the limit of $\mathbf{B}_{a,\delta}$ as $a \rightarrow \infty$. We refer to $\mathbf{B}_\delta^{\text{mod}}$ as the *model operator* for $\mathbf{B}_{a,\delta}$. In Lemma 4.2 we compute the kernel of

$$\text{ind } \mathbf{B}_\delta^{\text{mod}} = \text{ind}(D + \Phi). \quad (2.9)$$

Finally, in Proposition 5.3 we show that

$$\text{ind } \mathbf{B}_\delta^{\text{mod}} = \text{ind } \mathbf{B}_{a,\delta}. \quad (2.10)$$

Theorem 2.13 follows immediately from (2.8), (2.9), and (2.10).

2.15. The gluing formula. As a first application of Theorem 2.12 we prove the gluing formula, cf. Section 6.

Suppose that $(M, E, D + \Phi)$ is as in Subsection 2.3 and that Σ is a hypersurface in M . Under certain conditions (cf. Assumption 6.2), if one cuts M along Σ and converts it to a complete manifold without boundary by rescaling the metric, one gets a new triple $(M_\Sigma, E_\Sigma, D_\Sigma + \Phi_\Sigma)$, with $D_\Sigma + \Phi_\Sigma$ being a Callias-type and, hence, a Fredholm operator. Then the gluing formula asserts that

Theorem 2.16. *The operators $D + \Phi$ and $D_\Sigma + \Phi_\Sigma$ are cobordant. In particular,*

$$\text{ind}(D + \Phi) = \text{ind}(D_\Sigma + \Phi_\Sigma).$$

If M is partitioned into two relatively open submanifolds M_1 and M_2 by Σ , namely, $M = M_1 \cup \Sigma \cup M_2$, then the complete metric on M_Σ induces complete Riemannian metrics on M_1 and M_2 . Let E_i, D_i, Φ_i denote the restrictions of the graded vector bundle E_Σ and operators D_Σ, Φ_Σ to M_i ($i = 1, 2$). The above theorem implies the additivity of the index (cf. Corollary 6.10).

2.17. The relative index theorem. Section 7 is occupied with the second application of Theorem 2.12, which is a new proof of the well-known relative index theorem for Callias-type operators.

Consider two triples $(M_j, E_j, D_j + \Phi_j)$ as before ($j = 1, 2$). Suppose $M'_j \cup_{\Sigma_j} M''_j$ are partitions of M_j into relatively open submanifolds, where Σ_j are compact hypersurfaces.

Suppose there exist tubular neighborhoods $U(\Sigma_j)$ of Σ_j . We assume isomorphisms of structures between Σ_1 and Σ_2 , $U(\Sigma_1)$ and $U(\Sigma_2)$, $E_1|_{U(\Sigma_1)}$ and $E_2|_{U(\Sigma_2)}$. We also assume that Φ_j are invertible on $U(\Sigma_j)$, and that D_1, Φ_1 coincide with D_2, Φ_2 on $U(\Sigma_1) \simeq U(\Sigma_2)$ (cf. Assumption 7.2). Then we can cut M_j along Σ_j and use the isomorphism map to glue the pieces together interchanging M''_1 and M''_2 . In this way we obtain the manifolds

$$M_3 := M'_1 \cup_\Sigma M''_2, \quad M_4 := M'_2 \cup_\Sigma M''_1.$$

Similarly, we can do this cut-and-glue procedure to E_j to get new vector bundles E_3 over M_3 , E_4 over M_4 . After restricting D_j, Φ_j to each piece, we obtain Callias-type operators $D_3 + \Phi_3$ on M_3 , $D_4 + \Phi_4$ on M_4 , both having well-defined indexes.

For simplicity, we denote the Callias-type operators $D_j + \Phi_j$, $j = 1, 2, 3, 4$ by P_j . Then the relative index theorem can be stated as

Theorem 2.18. $\text{ind } P_1 + \text{ind } P_2 = \text{ind } P_3 + \text{ind } P_4$.

Our proof of the theorem involves the gluing formula. However, one has to do deformations to P_j , $j = 1, 2$ first in order to have (M_j, E_j, P_j) along with Σ_j satisfy the hypothesis of the gluing formula (cf. Subsections 7.5 and 7.8).

2.19. Callias-type index theorem. Using the relative index theorem, Anghel proved an important Callias-type index theorem in [2]. Since we give a new proof of the relative index theorem here, we also obtain a new proof of the Callias-type index theorem.

3. INDEX OF THE OPERATOR $\mathbf{B}_{a,\delta}$

In this section, we construct a family of operators $\mathbf{B}_{a,\delta}$ on $\tilde{W} := W \cup (\partial W \times [0, \infty))$, such that the index $\text{ind } \mathbf{B}_{a,\delta} = 0$. Later in Section 5, we show that $\text{ind } \mathbf{B}_{a,\delta} = \text{ind}(D + \Phi)$ for $a \gg 0$.

3.1. Construction of $\mathbf{B}_{a,\delta}$. Consider two anti-commuting actions (“left” and “right” action) of the Clifford algebra of \mathbb{R} on the exterior algebra $\wedge^\bullet \mathbb{C} = \wedge^0 \mathbb{C} \oplus \wedge^1 \mathbb{C}$ given by

$$c_L(t)\omega = t \wedge \omega - \iota_t \omega, \quad c_R(t)\omega = t \wedge \omega + \iota_t \omega. \quad (3.1)$$

We define $\tilde{W} := W \cup (M \times [0, \infty))$ as in Subsection 2.14 and extend the vector bundle F and the operators \tilde{D} , $\tilde{\Phi}$ to \tilde{W} in the natural way. Set $\tilde{F} := F \otimes \wedge^\bullet \mathbb{C}$ and consider the operator

$$B := \sqrt{-1}(\tilde{D} + \tilde{\Phi}) \otimes c_L(1) : C_0^\infty(\tilde{W}, \tilde{F}) \rightarrow C_0^\infty(\tilde{W}, \tilde{F}).$$

Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a smooth function with $f(t) = t$ for $t \geq 1$, and $f(t) = 0$ for $t \leq 1/2$. Consider the map $p : \tilde{W} \rightarrow \mathbb{R}$ such that $p(y, t) = f(t)$ for $(y, t) \in M \times (0, \infty)$ and $p(x) = 0$ for $x \in W$. For any $a \in \mathbb{R}$ and $\delta > 0$, define the operator

$$\mathbf{B}_{a,\delta} := B - 1 \otimes \delta \cdot c_R(p(x) - a). \quad (3.2)$$

Note that as a first order differential operator on the complete manifold \tilde{W} , the leading symbol of $\mathbf{B}_{a,\delta}$ is equal to $\sigma(\tilde{D})$. Hence it satisfies (2.3). We conclude that $\mathbf{B}_{a,\delta}$ is essentially self-adjoint by [13, Theorem 1.17].

Lemma 3.2. *Let $\Pi_i : \tilde{F} \rightarrow F \otimes \wedge^i \mathbb{C}$ ($i = 0, 1$) be the projections. Then*

$$\mathbf{B}_{a,\delta}^2 = (\tilde{D} + \tilde{\Phi})^2 \otimes 1 - \delta \cdot R + \delta^2 |p(x) - a|^2, \quad (3.3)$$

where R is a uniformly bounded bundle map whose restriction to W vanishes, and

$$R|_{M \times (1, \infty)} = \sqrt{-1} \gamma (\Pi_1 - \Pi_0), \quad \text{where} \quad \gamma|_{F^\pm} = \pm \sqrt{-1}. \quad (3.4)$$

Proof. Note first that $p(x) - a \equiv -a$ on W . Thus, since $c_R(-a)$ anti-commutes with B , we have $\mathbf{B}_{a,\delta}^2|_W = B^2|_W + \delta^2 a^2 = (\tilde{D} + \tilde{\Phi})^2 \otimes 1|_W + \delta^2 a^2$, which is (3.3).

Restricting $\mathbf{B}_{a,\delta}$ to the cylinder $M \times (0, \infty)$, we obtain

$$\mathbf{B}_{a,\delta}|_{M \times (0, \infty)} = \sqrt{-1}(D + \Phi) \otimes c_L(1) + \sqrt{-1} \gamma \otimes c_L(1) \partial_t - 1 \otimes \delta(f(t) - a) \cdot c_R(1).$$

Since c_L and c_R anti-commute, we get

$$\mathbf{B}_{a,\delta}^2|_{M \times (0, \infty)} = (\tilde{D} + \tilde{\Phi})^2 \otimes 1 - \sqrt{-1} \delta f' \gamma \otimes c_L(1) c_R(1) + \delta^2 |t - a|^2.$$

Since $c_L(1) c_R(1) = \Pi_1 - \Pi_0$, (3.3) and (3.4) follow with $R = f' \sqrt{-1} \gamma (\Pi_1 - \Pi_0)$. \square

3.3. Fredholmness of $\mathbf{B}_{a,\delta}$.

Lemma 3.4. *There exists a small enough δ , such that $\mathbf{B}_{a,\delta}$ is a Fredholm operator for every $a \in \mathbb{R}$.*

Proof. By [1, Theorem 2.1], it is enough to show that the operator $\mathbf{B}_{a,\delta}$ is invertible at infinity (cf. (2.4)). Since $\mathbf{B}_{a,\delta}$ is self-adjoint, (2.4) is equivalent to the fact that there exists a constant $\tilde{c} > 0$ and a compact $\tilde{K} \Subset \tilde{W}$ such that

$$(\mathbf{B}_{a,\delta}^2 s, s) \geq \tilde{c} \|s\|^2, \quad \text{for all } s \in L^2(\tilde{W}, \tilde{F}), \text{ supp}(s) \cap \tilde{K} = \emptyset, \quad (3.5)$$

where (\cdot, \cdot) denotes the inner product on $L^2(\tilde{W}, \tilde{F})$. Note that if we denote the bundle map

$$Q := (\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+) \otimes 1 - \delta \cdot R + \delta^2 |p(x) - a|^2,$$

then (3.3) can also be written as

$$\mathbf{B}_{a,\delta}^2 = \tilde{D}^2 \otimes 1 + Q.$$

Since \tilde{D}^2 is a non-negative operator on \tilde{W} , (3.5) can be reduced to

$$|Q(x)| \geq \tilde{c}, \quad \text{for all } x \in \tilde{W} \setminus \tilde{K}. \quad (3.6)$$

Since both $D + \Phi$ and $\tilde{D} + \tilde{\Phi}$ are Callias-type operators, there exist compact subsets $K \Subset M$, $K_W \Subset W$ and positive constants $c, c_W > 0$, such that

$$|(\Phi^2 + [D, \Phi]_+)(y)| \geq c, \quad \text{for all } y \in M \setminus K, \quad (3.7)$$

and

$$|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(x)| \geq c_W, \quad \text{for all } x \in W \setminus K_W. \quad (3.8)$$

Now consider $|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(y, t)|$ for $(y, t) \in M \times [0, \infty)$. Note that $\tilde{\Phi}$ is independent of t , and anti-commutes with $\gamma \partial_t$. So

$$[\tilde{D}, \tilde{\Phi}]_+ = (D + \gamma \partial_t) \tilde{\Phi} + \tilde{\Phi} (D + \gamma \partial_t) = D \tilde{\Phi} + \tilde{\Phi} D = [D, \tilde{\Phi}]_+.$$

Thus

$$|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(y, t)| = |(\Phi^2 + [D, \Phi]_+)(y)|,$$

which does not depend on t . From (3.7), we get

$$|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(y, t)| \geq c, \quad \text{for all } (y, t) \in (M \times [0, \infty)) \setminus (K \times [0, \infty)). \quad (3.9)$$

Furthermore, since K is compact, $\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+$ is bounded from below on $M \times [0, \infty)$.

Set $W_r := W \cup (M \times [0, r])$, $\tilde{K}_r := K_W \cup (K \times [0, r])$, $r > 0$ and $d_1 := \min\{c, c_W\}$. By (3.8) and (3.9),

$$|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(x)| \geq d_1, \quad \text{for all } x \in W_r \setminus \tilde{K}_r.$$

Since R is uniformly bounded on \tilde{W} , we can choose δ small enough such that

$$\delta \cdot \sup_{x \in \tilde{W}} |R(x)| \leq d_1/2.$$

So

$$|Q(x)| \geq \frac{d_1}{2}, \quad \text{for all } x \in W_r \setminus \tilde{K}_r. \quad (3.10)$$

Since $\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+$ has a uniform lower bound on $M \times [0, \infty)$, and $|p(y, t) - a|^2$ grows quadratically as $t \rightarrow \infty$, there exist $r = r(a, \delta)$ and $d_2 > 0$, such that

$$|Q(y, t)| \geq d_2, \quad \text{for all } (y, t) \in M \times [r, \infty). \quad (3.11)$$

Set $\tilde{K} := \tilde{K}_{r(a,\delta)}$ and $\tilde{c} := \min\{d_1/2, d_2\}$. Combining (3.10) and (3.11) yields (3.6). Therefore the lemma is proved. \square

3.5. Index of $\mathbf{B}_{a,\delta}$. From now on we fix δ which satisfies Lemma 3.4. Define a grading on the vector bundle $\tilde{F} = F \otimes \wedge^\bullet \mathbb{C}$ by

$$\tilde{F}^+ := F \otimes \wedge^0 \mathbb{C}, \quad \tilde{F}^- := F \otimes \wedge^1 \mathbb{C}, \quad (3.12)$$

and denote by $\mathbf{B}_{a,\delta}^\pm := \mathbf{B}_{a,\delta}|_{L^2(\tilde{W}, \tilde{F}^\pm)}$ the restrictions. We consider the index

$$\text{ind } \mathbf{B}_{a,\delta} := \dim \ker \mathbf{B}_{a,\delta}^+ - \dim \ker \mathbf{B}_{a,\delta}^-.$$

Lemma 3.6. *ind $\mathbf{B}_{a,\delta}$ is independent of a .*

Proof. Since for every $a, b \in \mathbb{R}$ the operator $\mathbf{B}_{b,\delta} - \mathbf{B}_{a,\delta} = 1 \otimes \delta \cdot c_R(b - a)$ is bounded and depends continuously on $b - a \in \mathbb{R}$, the lemma follows from the stability of the index of a Fredholm operator. \square

Lemma 3.7. *ind $\mathbf{B}_{a,\delta} = 0$ for all $a \in \mathbb{R}$.*

Proof. By Lemma 3.6, it suffices to prove this result for a particular value of a . If a is a negative number such that $a^2 > \sup_{x \in \tilde{W}} |R(x)|/\delta$, then $\mathbf{B}_{a,\delta}^2 > 0$ by (3.3), so that $\ker \mathbf{B}_{a,\delta} = 0 = \text{ind } \mathbf{B}_{a,\delta}$. \square

4. THE MODEL OPERATOR

When a is large, all the sections $s \in \ker \mathbf{B}_{a,\delta}$ are concentrated on the cylinder $M \times [0, \infty)$ near $M \times \{a\}$. Thus index of $\mathbf{B}_{a,\delta}$ is related to the index of a certain operator on $M \times \mathbb{R}$, whose restriction to a neighborhood of $M \times \{a\}$ in \tilde{W} is an approximation of the restriction of $\mathbf{B}_{a,\delta}$ to the neighborhood of $M \times \{a\}$ in $M \times \mathbb{R}$. We call this operator the *model operator* for $\mathbf{B}_{a,\delta}$ and denote it by $\mathbf{B}_\delta^{\text{mod}}$. In this section we construct the model operator and show that $\text{ind } \mathbf{B}_\delta^{\text{mod}} = \text{ind}(D + \Phi)$. In the next section we show that its index is equal to the index of $\mathbf{B}_{a,\delta}$.

4.1. The operator $\mathbf{B}_\delta^{\text{mod}}$. Consider the lift of the bundle $E = E^+ \oplus E^-$ to the cylinder $M \times \mathbb{R}$, which will still be denoted by $E = E^+ \oplus E^-$.

Consider the vector bundle $\tilde{F}^{\text{mod}} = (E^+ \oplus E^-) \otimes \wedge^\bullet \mathbb{C}$ over $M \times \mathbb{R}$ and the operator

$$\mathbf{B}_\delta^{\text{mod}} : L^2(M \times \mathbb{R}, \tilde{F}^{\text{mod}}) \rightarrow L^2(M \times \mathbb{R}, \tilde{F}^{\text{mod}})$$

defined by

$$\mathbf{B}_\delta^{\text{mod}} := \sqrt{-1}(D + \Phi) \otimes c_L(1) + \sqrt{-1}\gamma \otimes c_L(1)\partial_t - 1 \otimes \delta \cdot c_R(t), \quad (4.1)$$

where t is the coordinate along the axis of the cylinder, $\gamma|_{E^\pm} = \pm\sqrt{-1}$, and δ is fixed with the same value as in Subsection 3.5. The operator $\mathbf{B}_\delta^{\text{mod}}$ satisfies Assumption 2.4 as well and, hence, is self-adjoint. Like in Lemma 3.2, we have

$$(\mathbf{B}_\delta^{\text{mod}})^2 = (D + \gamma\partial_t + \Phi)^2 \otimes 1 - \sqrt{-1}\delta\gamma(\Pi_1 - \Pi_0) + \delta^2 t^2.$$

Then by the same argument as in the proof of Lemma 3.4, $\mathbf{B}_\delta^{\text{mod}}$ is a Fredholm operator.

Clearly, the restrictions of \tilde{F}^{mod} and \tilde{F} to the cylinder $M \times (1, \infty)$ are isomorphic. We give \tilde{F}^{mod} grading similar to (3.12),

$$\tilde{F}_+^{\text{mod}} := E \otimes \wedge^0 \mathbb{C}, \quad \tilde{F}_-^{\text{mod}} := E \otimes \wedge^1 \mathbb{C}.$$

Set

$$\text{ind } \mathbf{B}_\delta^{\text{mod}} := \dim \ker(\mathbf{B}_\delta^{\text{mod}})_+ - \dim \ker(\mathbf{B}_\delta^{\text{mod}})_-,$$

where $(\mathbf{B}_\delta^{\text{mod}})_\pm := \mathbf{B}_\delta^{\text{mod}}|_{L^2(\tilde{W}, \tilde{F}_\pm^{\text{mod}})}$.

Lemma 4.2. *The space $\ker \mathbf{B}_\delta^{\text{mod}}$ is isomorphic (as a graded space) to $\ker(D + \Phi)$. In particular,*

$$\text{ind } \mathbf{B}_\delta^{\text{mod}} = \text{ind}(D + \Phi). \quad (4.2)$$

Proof. The space $L^2(M \times \mathbb{R}, E^\pm \otimes \wedge^\bullet \mathbb{C})$ decomposes into a tensor product

$$L^2(M \times \mathbb{R}, E^\pm \otimes \wedge^\bullet \mathbb{C}) = L^2(M, E^\pm) \otimes L^2(\mathbb{R}, \wedge^\bullet \mathbb{C}).$$

From (4.1) it follows that with respect to this decomposition we have

$$(\mathbf{B}_\delta^{\text{mod}})^2|_{L^2(M \times \mathbb{R}, E^\pm \otimes \wedge^\bullet \mathbb{C})} = (D + \Phi)^2 \otimes 1 + 1 \otimes (-\partial_{tt} \pm \delta(\Pi_1 - \Pi_0) + \delta^2 t^2).$$

Notice that both summands on the right hand side are non-negative.

The space $\ker(-\partial_{tt} + \Pi_1 - \Pi_0 + t^2) \subset L^2(\mathbb{R}, \wedge^\bullet \mathbb{C})$ is one-dimensional and is spanned by

$$\alpha^+(t) = e^{-\delta t^2/2} \in L^2(\mathbb{R}, \wedge^0 \mathbb{C}).$$

Similarly, the space $\ker(-\partial_{tt} + \Pi_0 - \Pi_1 + t^2)$ is one-dimensional and is spanned by

$$\alpha^-(t) = e^{-\delta t^2/2} ds \in L^2(\mathbb{R}, \wedge^1 \mathbb{C}).$$

It follows that

$$\ker(\mathbf{B}_\delta^{\text{mod}})^2|_{L^2(M \times \mathbb{R}, E^\pm \otimes \wedge^\bullet \mathbb{C})} \simeq \{\sigma \otimes \alpha^\pm(t) : \sigma \in \ker(D + \Phi)^2|_{L^2(M, E^\pm)}\}.$$

□

4.3. The operator $\mathbf{B}_{a,\delta}^{\text{mod}}$. Let

$$T_a : M \times \mathbb{R} \rightarrow M \times \mathbb{R}, \quad T_a(x, t) = (x, t + a)$$

be the translation and consider the pull-back map

$$T_a^* : L^2(M \times \mathbb{R}, \tilde{F}^{\text{mod}}) \rightarrow L^2(M \times \mathbb{R}, \tilde{F}^{\text{mod}}).$$

Set

$$\mathbf{B}_{a,\delta}^{\text{mod}} := T_{-a}^* \circ \mathbf{B}_\delta^{\text{mod}} \circ T_a^* = \sqrt{-1}(D + \Phi) \otimes c_L(1) + \sqrt{-1}\gamma \otimes c_L(1)\partial_t - 1 \otimes \delta \cdot c_R(t - a).$$

Then

$$\dim \ker(\mathbf{B}_{a,\delta}^{\text{mod}})_\pm = \dim \ker(\mathbf{B}_\delta^{\text{mod}})_\pm \quad (4.3)$$

for all $a \in \mathbb{R}$.

5. PROOF OF THEOREM 2.13

In this section, we finish the proof of the cobordism invariance of the Callias-type index by showing that $\text{ind } \mathbf{B}_{a,\delta} = \text{ind } \mathbf{B}_\delta^{\text{mod}}$. Since δ is fixed throughout the section, we omit it from the notation, and write $\mathbf{B}_a, \mathbf{B}^{\text{mod}}$ and $\mathbf{B}_a^{\text{mod}}$ for $\mathbf{B}_{a,\delta}, \mathbf{B}_\delta^{\text{mod}}$ and $\mathbf{B}_{a,\delta}^{\text{mod}}$, respectively.

5.1. The spectral counting function. For a self-adjoint operator P and a real number λ , we denote by $N(\lambda, P)$ the number of eigenvalues of P not exceeding λ (counting with multiplicities). If the intersection of the continuum spectrum of P with the set $(-\infty, \lambda]$ is not empty, then we set $N(\lambda, P) = \infty$.

Let \mathbf{B}_a^\pm denote the restrictions of \mathbf{B}_a to the spaces $L^2(\tilde{W}, \tilde{F}^\pm)$ and let $\mathbf{B}_\pm^{\text{mod}}, \mathbf{B}_{a,\pm}^{\text{mod}}$ denote the restrictions of $\mathbf{B}^{\text{mod}}, \mathbf{B}_a^{\text{mod}}$ to the spaces $L^2(M \times \mathbb{R}, \tilde{F}_\pm^{\text{mod}})$.

Since the operator \mathbf{B}^{mod} is self-adjoint, by von Neumann's theorem (cf. [18, Theorem X.25]), the operators $(\mathbf{B}^{\text{mod}})_\pm^2 = \mathbf{B}_\mp^{\text{mod}} \mathbf{B}_\pm^{\text{mod}} = (\mathbf{B}_\pm^{\text{mod}})^* \mathbf{B}_\pm^{\text{mod}}$ are also self-adjoint. Since the operators $(\mathbf{B}^{\text{mod}})_\pm^2$ are Fredholm, they have smallest non-zero elements of the spectra, denoted by λ_\pm .

Lemma 5.2. $\lambda_+ = \lambda_-$.

Proof. Since $(\mathbf{B}^{\text{mod}})_+^2 = \mathbf{B}_-^{\text{mod}} \mathbf{B}_+^{\text{mod}}, (\mathbf{B}^{\text{mod}})_-^2 = \mathbf{B}_+^{\text{mod}} \mathbf{B}_-^{\text{mod}}$, by [15, Theorem 1.1], their spectra satisfy

$$\sigma((\mathbf{B}^{\text{mod}})_+^2) \setminus \{0\} = \sigma((\mathbf{B}^{\text{mod}})_-^2) \setminus \{0\}.$$

In particular, $\lambda_+ = \lambda_-$. □

From now on, we set

$$\lambda := \lambda_+ = \lambda_-.$$

Proposition 5.3. *For any $0 < \epsilon < \lambda$, there exists $A = A(\epsilon, \delta, p) > 0$, such that*

$$N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm) = \dim \ker(\mathbf{B}^{\text{mod}})_\pm^2, \quad \text{for all } a > A, \quad (5.1)$$

where $\delta > 0$, $p : \tilde{W} \rightarrow \mathbb{R}$ are as in Subsection 3.1. In particular,

$$\text{ind } \mathbf{B}^{\text{mod}} = N(\lambda - \epsilon, (\mathbf{B}_a^2)^+) - N(\lambda - \epsilon, (\mathbf{B}_a^2)^-). \quad (5.2)$$

Before proving this proposition we show how it implies Theorem 2.13.

5.4. Proof of Theorem 2.13. By Proposition 5.3, $N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm) < \infty$. Let

$$V_{\epsilon,a}^\pm \subset L^2(\tilde{W}, \tilde{F}^\pm)$$

denote the vector spaces spanned by the eigenvectors of the operators $(\mathbf{B}_a^2)^\pm$ with eigenvalues within $(0, \lambda - \epsilon]$. Then $\dim V_{\epsilon,a}^\pm < \infty$ and the restrictions of the operators \mathbf{B}_a^\pm to $V_{\epsilon,a}^\pm$ define bijections

$$\mathbf{B}_a^\pm : V_{\epsilon,a}^\pm \longrightarrow V_{\epsilon,a}^\mp.$$

Hence,

$$\dim V_{\epsilon,a}^+ = \dim V_{\epsilon,a}^-.$$

Thus

$$\begin{aligned} N(\lambda - \epsilon, (\mathbf{B}_a^2)^+) - N(\lambda - \epsilon, (\mathbf{B}_a^2)^-) &= (\dim \ker \mathbf{B}_a^+ + \dim V_{\epsilon,a}^+) - (\dim \ker \mathbf{B}_a^- + \dim V_{\epsilon,a}^-) \\ &= \dim \ker \mathbf{B}_a^+ - \dim \ker \mathbf{B}_a^- = \text{ind } \mathbf{B}_a. \end{aligned}$$

From Proposition 5.3 we now obtain

$$\text{ind } \mathbf{B}_a = \text{ind } \mathbf{B}^{\text{mod}}$$

and Theorem 2.13 follows from Lemma 3.7 and 4.2. \square

The rest of this section is occupied with the proof of Proposition 5.3.

5.5. Estimate from above on $N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm)$. First we show that

$$N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm) \leq \dim \ker \mathbf{B}_\pm^{\text{mod}}. \quad (5.3)$$

This is done through the following techniques.

5.6. The IMS localization. Let $j, \hat{j} : \mathbb{R} \rightarrow [0, 1]$ be smooth functions such that $j^2 + \hat{j}^2 \equiv 1$ and $j(t) = 1$ for $t \geq 3$, while $j(t) = 0$ for $t \leq 2$. Set $j_a(t) = j(a^{-1/2}t)$, $\hat{j}_a(t) = \hat{j}(a^{-1/2}t)$. Now we view them as functions on the cylinder $M \times [0, \infty)$ (whose points are written as (y, t)). Similarly, we still use the same notations $j_a(x) = j(a^{-1/2}p(x))$, $\hat{j}_a(x) = \hat{j}(a^{-1/2}p(x))$ to denote the functions on \tilde{W} , where $p(x)$ is defined in Subsection 3.1.

We use the following version of the IMS localization, cf. [20, §3], [5, Lemma 4.5]¹

Lemma 5.7. *The following operator identity holds:*

$$\mathbf{B}_a^2 = \hat{j}_a \mathbf{B}_a^2 \hat{j}_a + j_a \mathbf{B}_a^2 j_a + \frac{1}{2}[\hat{j}_a, [\hat{j}_a, \mathbf{B}_a^2]] + \frac{1}{2}[j_a, [j_a, \mathbf{B}_a^2]]. \quad (5.4)$$

Now we estimate each summand on the right-hand side of (5.4).

¹The abbreviation IMS is formed by the initials of the surnames of R. Ismagilov, J. Morgan, I. Sigal and B. Simon.

Lemma 5.8. *There exists $A = A(\delta, p) > 0$ such that*

$$\hat{j}_a \mathbf{B}_a^2 \hat{j}_a \geq \frac{\delta^2 a^2}{8} \hat{j}_a^2$$

for all $a > A$.

Proof. Note that if $x \in \text{supp } \hat{j}_a$, then $p(x) \leq 3a^{1/2}$. Hence for $a > 36$, we have

$$\hat{j}_a^2 |p(x) - a|^2 \geq \frac{a^2}{4} \hat{j}_a^2.$$

Set $A = \max\{36, 4\delta^{1/2} \sup_{x \in \tilde{W}} \|R(x)\|^{1/2}\}$ and let $a > A$. By Lemma 3.2,

$$\hat{j}_a \mathbf{B}_a^2 \hat{j}_a \geq \hat{j}_a^2 \delta^2 |p(x) - a|^2 - \hat{j}_a (\delta \cdot R) \hat{j}_a \geq \frac{\delta^2 a^2}{8} \hat{j}_a^2.$$

□

5.9. Let $\Pi_a : L^2(M \times \mathbb{R}, \tilde{F}^{\text{mod}}) \rightarrow \ker \mathbf{B}_a^{\text{mod}}$ be the orthogonal projection and Π_a^\pm be the restrictions of Π_a to the spaces $L^2(M \times \mathbb{R}, \tilde{F}_\pm^{\text{mod}})$. Then Π_a^\pm are finite rank operators and their ranks are $\dim \ker \mathbf{B}_{a,\pm}^{\text{mod}}$, which are equal to $\dim \ker \mathbf{B}_\pm^{\text{mod}}$ by (4.3). Since $(\mathbf{B}_a^{\text{mod}})_\pm^2$ are nonnegative operators, it's clear that

$$(\mathbf{B}_a^{\text{mod}})_\pm^2 + \lambda \Pi_a^\pm \geq \lambda. \quad (5.5)$$

Observe that $\text{supp } j_a$ in $M \times \mathbb{R}$ is a subset of $M \times [0, \infty)$. It's a subset of $\tilde{W} = W \cup (M \times [0, \infty))$ as well. So we can consider $j_a \Pi_a j_a$ and $j_a \mathbf{B}_a^{\text{mod}} j_a$ as operators on \tilde{W} . Then $j_a \mathbf{B}_a^2 j_a = j_a (\mathbf{B}_a^{\text{mod}})^2 j_a$. Hence, (5.5) implies the following.

Lemma 5.10. $j_a (\mathbf{B}_a^2)^\pm j_a + \lambda j_a \Pi_a^\pm j_a \geq \lambda j_a^2$, $\text{rank } j_a \Pi_a^\pm j_a \leq \dim \ker \mathbf{B}_\pm^{\text{mod}}$.

The next lemma estimates the last two summands on the right-hand side of (5.4).

Lemma 5.11. *Let $C = 2 \max \{ \max\{|j'(t)|^2, |\hat{j}'(t)|^2\} : t \in \mathbb{R} \}$. Then*

$$\|[j_a, [j_a, \mathbf{B}_a^2]]\| \leq C a^{-1}, \quad \|[\hat{j}_a, [\hat{j}_a, \mathbf{B}_a^2]]\| \leq C a^{-1} \quad \text{for all } a > 0. \quad (5.6)$$

Proof. By Lemma 3.2, we get

$$\|[j_a, [j_a, \mathbf{B}_a^2]]\| = 2|j'_a(t)|^2 = 2a^{-1}|j'(a^{-1/2}t)|^2,$$

$$\|[\hat{j}_a, [\hat{j}_a, \mathbf{B}_a^2]]\| = 2|\hat{j}'_a(t)|^2 = 2a^{-1}|\hat{j}'(a^{-1/2}t)|^2.$$

Then (5.6) follows immediately. □

Since λ is fixed, combining Lemma 5.7, 5.8, 5.10 and 5.11, we obtain

Corollary 5.12. *For any $\epsilon > 0$, there exists $A = A(\epsilon, \delta, p) > 0$ such that, for all $a > A$,*

$$(\mathbf{B}_a^2)^\pm + \lambda j_a \Pi_a^\pm j_a \geq \lambda - \epsilon, \quad \text{rank } j_a \Pi_a^\pm j_a \leq \dim \ker \mathbf{B}_\pm^{\text{mod}}. \quad (5.7)$$

The estimate (5.3) now follows from Corollary 5.12 and the following result, [19, p. 270]:

Lemma 5.13. *Assume that P, Q are self-adjoint operators on a Hilbert space such that $\text{rank } Q \leq k$ and there exists $\mu > 0$ such that $\langle (P + Q)u, u \rangle \geq \mu \langle u, u \rangle$ for any $u \in \text{Dom}(P)$. Then $N(\mu - \epsilon, P) \leq k$ for any $\epsilon > 0$.*

5.14. **Estimate from below on $N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm)$.** Now it remains to prove that

$$N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm) \geq \dim \ker \mathbf{B}_\pm^{\text{mod}} = \dim \ker \mathbf{B}_{a,\pm}^{\text{mod}}. \quad (5.8)$$

By (5.3), $N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm)$ are finite for a large enough. Under this circumstance, let $\tilde{V}_{\epsilon,a}^\pm \subset L^2(\tilde{W}, \tilde{F}^\pm)$ denote the vector spaces spanned by the eigenvectors of the operators $(\mathbf{B}_a^2)^\pm$ for eigenvalues within $[0, \lambda - \epsilon]$. Let $\Theta_{\epsilon,a}^\pm : L^2(\tilde{W}, \tilde{F}^\pm) \rightarrow \tilde{V}_{\epsilon,a}^\pm$ be the orthogonal projections. Then $\text{rank } \Theta_{\epsilon,a}^\pm = N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm)$. As in Subsection 5.9, we can consider $j_a \Theta_{\epsilon,a}^\pm j_a$ as operators on $L^2(M \times \mathbb{R}, \tilde{F}_\pm^{\text{mod}})$. Then the same argument as in the proof of Corollary 5.12 works here and we have

Lemma 5.15. *For any $\epsilon > 0$, there exists $A = A(\epsilon, \delta) > 0$ such that, for all $a > A$,*

$$(\mathbf{B}_a^{\text{mod}})_\pm^2 + \lambda j_a \Theta_a^\pm j_a \geq \lambda - \epsilon, \quad \text{rank } j_a \Theta_a^\pm j_a \leq \dim N(\lambda - \epsilon, (\mathbf{B}_a^2)^\pm). \quad (5.9)$$

Similarly, the estimate (5.8) follows from Lemma 5.15 and 5.13.

Now the proof of Proposition 5.3 and, hence, of Theorem 2.13 is complete. \square

6. THE GLUING FORMULA

Our first application of Theorem 2.12 is the gluing formula. If we cut a complete manifold along a hypersurface Σ , we obtain a manifold with boundary. By rescaling the metric near the boundary, we may convert it to a complete manifold without boundary. In this section, we show that the index of a Callias-type operator is invariant under this type of surgery. In particular, if M is partitioned into two pieces M_1 and M_2 by Σ , we see that the index on M is equal to the sum of the indexes on M_1 and M_2 . In other words, the index is additive.

6.1. **The surgery.** Let $(M, E, D + \Phi)$ be as in Subsection 2.3 with $\dim M = n$, where

$$D : C_0^\infty(M, E^\pm) \rightarrow C_0^\infty(M, E^\mp)$$

satisfies Assumption 2.4 and $D + \Phi$ is a Callias-type operator. Suppose $\Sigma \subset M$ is a smooth hypersurface. For simplicity, we assume that Σ is compact.

Throughout this section we make the following assumption.

Assumption 6.2. There exist a compact set $K \Subset M$ and two constants $c_1, c_2 > 0$ such that

(i) for all $x \in M \setminus K$,

$$|(\Phi^2 + [D, \Phi]_+)(x)| \geq c_1, \quad |\Phi^2(x)| \geq c_2;$$

(ii) $\Sigma \subset M \setminus K$, which indicates that K is still a compact subset of $M_\Sigma := M \setminus \Sigma$.

We denote by E_Σ the restriction of the graded vector bundle E to M_Σ . Let g denote the Riemannian metric on M . By a rescaling of g near Σ , one can obtain a complete Riemannian metric on M_Σ and a Callias-type operator $D_\Sigma + \Phi_\Sigma$ on M_Σ . It follows from the cobordism invariance of the index (cf. Theorem 2.12) that the index of $D_\Sigma + \Phi_\Sigma$ is independent of the choice of a rescaling.

6.3. **A rescaling of the metric.** We now present one of the possible constructions of a complete metric on M_Σ .

Let $\tau : M \rightarrow [-1, 1]$ be a smooth function, such that $\tau^{-1}(0) = \Sigma$ and τ is regular at Σ . Set $\alpha(x) = (\tau(x))^2$. Define the metric g_Σ on M_Σ by

$$g_\Sigma := \frac{1}{\alpha(x)^2} g. \quad (6.1)$$

This makes (M_Σ, g_Σ) a complete Riemannian manifold.

Let $d\text{vol}_g(x)$ and $d\text{vol}_{g_\Sigma}(x)$ denote the canonical volume forms on (M, g) and (M_Σ, g_Σ) , respectively. It's easy to see that $d\text{vol}_{g_\Sigma}(x) = \frac{1}{\alpha(x)^n} d\text{vol}_g(x)$. So the L^2 -inner product on $L^2(M_\Sigma, E_\Sigma)$ becomes

$$(s_1, s_2)_\Sigma = \int_{M_\Sigma} \langle s_1(x), s_2(x) \rangle_{(E_\Sigma)_x} \frac{1}{\alpha(x)^n} d\text{vol}_g(x). \quad (6.2)$$

6.4. The Callias-type operator on (M_Σ, g_Σ) . In order to get a natural Callias-type operator acting on $C_0^\infty(M_\Sigma, E_\Sigma)$ we set

$$\Phi_\Sigma := \Phi|_{M_\Sigma},$$

and

$$D_\Sigma(x)(s) := \alpha(x)^{\frac{n+1}{2}} D(x)(\alpha(x)^{-\frac{n-1}{2}} s), \quad \text{for all } x \in M_\Sigma, s \in C_0^\infty(M_\Sigma, E_\Sigma). \quad (6.3)$$

It's easy to check that

$$\sigma(D_\Sigma)(x, \xi) = \alpha(x) \sigma(D)(x, \xi)$$

So D_Σ also satisfies Assumption 2.4. Thus D_Σ and $D_\Sigma + \Phi_\Sigma : C_0^\infty(M_\Sigma, E_\Sigma^\pm) \rightarrow C_0^\infty(M_\Sigma, E_\Sigma^\mp)$ are still \mathbb{Z}_2 -graded first-order elliptic operators, which are essentially self-adjoint with respect to the L^2 -inner product defined by (6.2).

Remark 6.5. If E is a Clifford bundle with respect to g , and D is the Dirac operator, then E_Σ also has a Clifford structure with respect to g_Σ , and D_Σ defined by (6.3) is precisely the associated Dirac operator.

Lemma 6.6. $D_\Sigma + \Phi_\Sigma$ is a Callias-type operator, and, hence, is Fredholm.

Proof. Since $[D, \Phi]_+$ is a bundle map, a direct computation gives that

$$\begin{aligned} [D_\Sigma, \Phi_\Sigma]_+(s) &= D_\Sigma \Phi_\Sigma(s) + \Phi_\Sigma D_\Sigma(s) \\ &= \alpha^{\frac{n+1}{2}} D(\alpha^{-\frac{n-1}{2}} \Phi(s)) + \Phi(\alpha^{\frac{n+1}{2}} D(\alpha^{-\frac{n-1}{2}} s)) \\ &= \alpha^{\frac{n+1}{2}} D(\Phi(\alpha^{-\frac{n-1}{2}} s)) + \alpha^{\frac{n+1}{2}} \Phi(D(\alpha^{-\frac{n-1}{2}} s)) \\ &= \alpha^{\frac{n+1}{2}} [D, \Phi]_+(\alpha^{-\frac{n-1}{2}} s) = \alpha [D, \Phi]_+(s). \end{aligned}$$

So $[D_\Sigma, \Phi_\Sigma]_+$ is a bundle map as well. Then

$$\Phi_\Sigma^2 + [D_\Sigma, \Phi_\Sigma]_+ = (\Phi^2 + \alpha[D, \Phi]_+)|_{M_\Sigma} = ((1 - \alpha)\Phi^2 + \alpha(\Phi^2 + [D, \Phi]_+))|_{M_\Sigma}.$$

Note that $\alpha(x) \in [0, 1]$, by Assumption 6.2,

$$|(\Phi_\Sigma^2 + [D_\Sigma, \Phi_\Sigma]_+)(x)| \geq c, \quad \text{for all } x \in M_\Sigma \setminus K,$$

where $c := \min\{c_1, c_2\}$. Thus $D_\Sigma + \Phi_\Sigma$ is a Callias-type operator and, hence, is Fredholm by Lemma 2.6. \square

It follows from the above lemma that the index $\text{ind}(D_\Sigma + \Phi_\Sigma)$ is well defined.

6.7. The gluing formula. Under the above setting, there are two well-defined indexes $\text{ind}(D + \Phi)$ and $\text{ind}(D_\Sigma + \Phi_\Sigma)$.

Theorem 6.8. *The operators $D + \Phi$ and $D_\Sigma + \Phi_\Sigma$ are cobordant in the sense of Definition 2.8. In particular,*

$$\text{ind}(D + \Phi) = \text{ind}(D_\Sigma + \Phi_\Sigma).$$

We refer to Theorem 6.8 as a *gluing formula*, meaning that M is obtained from M_Σ by gluing along Σ .

Proof. The goal is to find a triple $(W, F, \tilde{D} + \tilde{\Phi})$, such that it is the cobordism between $(M, E, D + \Phi)$ and $(M_\Sigma, E_\Sigma, D_\Sigma + \Phi_\Sigma)$ and then apply Theorem 2.12.

Consider

$$W := \left\{ (x, t) \in M \times [0, \infty) : t \leq \frac{1}{\alpha(x)} + 1 \right\}.$$

Then W is a noncompact manifold whose boundary is diffeomorphic to the disjoint union of $M \simeq M \times \{0\}$ and $M \setminus \Sigma \simeq \{(x, \frac{1}{\alpha(x)} + 1)\}$. Essentially, W is the required cobordism. However, to be precise, we need to define a complete Riemannian metric g^W on W , such that condition (i) of Definition 2.8 is fulfilled.

Let $\beta : W \rightarrow [0, 1]$ be a smooth function such that $\beta(x, t) = 1$ for $0 \leq t \leq 1/2$, $\beta(x, t) > 0$ for $1/2 < t < 1/\alpha(x) + 1/2$ and $\beta(x, t) = \alpha(x)$ for $1/\alpha(x) + 1/2 \leq t \leq 1/\alpha(x) + 1$. Define the metric g^W on W by

$$g^W((\xi_1, \eta_1), (\xi_2, \eta_2)) := \frac{1}{\beta(x, t)^2} g(\xi_1, \xi_2) + \eta_1 \eta_2,$$

where $(\xi_1, \eta_1), (\xi_2, \eta_2) \in T_x M \oplus \mathbb{R} \simeq T_{(x, t)} W$. Then g^W is a complete metric.

Consider the neighborhood

$$U := \{(x, t) : 0 \leq t < 1/3\} \sqcup \left\{ (x, t) : \frac{1}{\alpha(x)} + \frac{2}{3} < t \leq \frac{1}{\alpha(x)} + 1 \right\}$$

of ∂W . Then define a map $\phi : (M \times [0, 1/3)) \sqcup (M_\Sigma \times (-1/3, 0]) \rightarrow U$ by the formulas

$$\begin{aligned} \phi(x, t) &:= (x, t), & x \in M, 0 \leq t < 1/3, \\ \phi(x, t) &:= \left(x, \frac{1}{\alpha(x)} + 1 + t \right), & x \in M_\Sigma, -1/3 < t \leq 0. \end{aligned}$$

It's easy to see that ϕ is a metric-preserving diffeomorphism, satisfying condition (i) of Definition 2.8.

Let $\pi : M \times [0, \infty) \rightarrow M$ be the projection. Then the pull-back $\pi^* E$ is a vector bundle over $M \times [0, \infty)$. Define

$$F := \pi^* E|_W.$$

So F is a vector bundle over W , whose restriction to the first part of U is isomorphic to the lift of E over $M \times [0, 1/3)$ and whose restriction to the second part of U is isomorphic to the lift of E_Σ over $M_\Sigma \times (-1/3, 0]$. Hence condition (ii) of Definition 2.8 is fulfilled. Note that here we can give F a natural grading which is compatible with that on E and E_Σ :

$$F^+ := \pi^* E^+|_W, \quad F^- := \pi^* E^-|_W.$$

We still use D and Φ to denote the lifts of D and Φ to $M \times [0, \infty)$. Now we define

$$\tilde{D}, \tilde{\Phi} : C_0^\infty(W, F) \rightarrow C_0^\infty(W, F)$$

by

$$\begin{aligned} \tilde{D}(\tilde{s}) &:= (\beta^{\frac{n+1}{2}} D|_W)(\beta^{-\frac{n-1}{2}} \tilde{s}) + \gamma \partial_t(\tilde{s}), & \text{for all } \tilde{s} \in C_0^\infty(W, F), \\ \tilde{\Phi} &:= \Phi|_W, \end{aligned} \tag{6.4}$$

where $\gamma|_{F^\pm} = \pm\sqrt{-1}$. Then $\sigma(\tilde{D}) = \beta(\sigma(D|_W)) + \sigma(\gamma \partial_t)$. Since β lies in $[0, 1]$, \tilde{D} satisfies Assumption 2.4. Moreover, $\tilde{D} + \tilde{\Phi}$ takes the form $D + \gamma \partial_t + \Phi$ on one end $M \times [0, 1/3)$ and the form $D_\Sigma + \gamma \partial_t + \Phi_\Sigma$ on the other end $M_\Sigma \times (-1/3, 0]$. So $\tilde{D} + \tilde{\Phi}$ has exactly the form required in condition (iii) of Definition 2.8.

It remains to verify that $\tilde{D} + \tilde{\Phi}$ is a Callias-type operator. Note that $\gamma \partial_t$ anti-commutes with $\tilde{\Phi}$, by the same computation as in the proof of Lemma 6.6, we have

$$[\tilde{D}, \tilde{\Phi}]_+ = \beta[D, \Phi]_+|_W$$

is a bundle map. And

$$\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+ = ((1 - \beta)\Phi^2 + \beta(\Phi^2 + [D, \Phi]_+))|_W.$$

By Assumption 6.2,

$$\tilde{K} := \left\{ (x, t) \in M \times [0, \infty) : x \in K, t \leq \frac{1}{\alpha(x)} + 1 \right\}$$

is a compact subset of W , and

$$|(\Phi^2 + [D, \Phi]_+)(x, t)| \geq c_1, \quad |\Phi^2(x, t)| \geq c_2, \quad \text{for all } (x, t) \in W \setminus \tilde{K}.$$

Again by $\beta \in [0, 1]$, we get

$$|(\tilde{\Phi}^2 + [\tilde{D}, \tilde{\Phi}]_+)(x, t)| \geq c, \quad \text{for all } (x, t) \in W \setminus \tilde{K},$$

where c is the same as in the proof of Lemma 6.6. Thus $\tilde{D} + \tilde{\Phi}$ is a Callias-type operator, and condition (iii) of Definition 2.8 is also fulfilled.

Therefore, $(W, F, \tilde{D} + \tilde{\Phi})$ is a cobordism between (M, E, D) and $(M_\Sigma, E_\Sigma, D_\Sigma + \Phi_\Sigma)$, and by Theorem 2.12, $\text{ind}(D + \Phi) = \text{ind}(D_\Sigma + \Phi_\Sigma)$. \square

6.9. The additivity of the index. Suppose that M is partitioned into two relatively open submanifolds M_1 and M_2 by Σ , so that $M_\Sigma = M_1 \sqcup M_2$. The metric g_Σ induces complete Riemannian metrics g_{M_1}, g_{M_2} on M_1 and M_2 , respectively. Let E_i, D_i, Φ_i denote the restrictions of the graded vector bundle E_Σ and operators D_Σ, Φ_Σ to M_i ($i = 1, 2$). Then Theorem 6.8 implies the following corollary.

Corollary 6.10. $\text{ind}(D + \Phi) = \text{ind}(D_1 + \Phi_1) + \text{ind}(D_2 + \Phi_2)$.

Thus we see that the index is “additive”.

7. RELATIVE INDEX THEOREM FOR CALLIAS-TYPE OPERATORS

As a second application of Theorem 2.12, and also as an application of Corollary 6.10, we give a new proof of the relative index theorem for Callias-type operators. There are several different forms of relative index theorem. In this paper we follow the approach of [9].

7.1. Setting. Let $(M_j, E_j, D_j + \Phi_j)$, $j = 1, 2$ be two triples of complete Riemannian manifold endowed with a \mathbb{Z}_2 -graded Hermitian vector bundle and with the associated Callias-type operator acting on the compactly supported smooth sections of the bundle. Suppose they satisfy Assumption 6.2.(i) of Subsection 6.1. In particular, the indexes $\text{ind}(D_j + \Phi_j)$, $j = 1, 2$ are well-defined.

Suppose $M'_j \cup_{\Sigma_j} M''_j$ are partitions of M_j into relatively open submanifolds, where Σ_j are compact hypersurfaces. We make the following assumption.

Assumption 7.2. There exist tubular neighborhoods $U(\Sigma_1), U(\Sigma_2)$ of Σ_1 and Σ_2 such that:

(i) there is a commutative diagram of isometric diffeomorphisms

$$\begin{array}{ccccc} \psi & : & E_1|_{U(\Sigma_1)} & \rightarrow & E_2|_{U(\Sigma_2)} \\ & & \downarrow & & \downarrow \\ \phi & : & U(\Sigma_1) & \rightarrow & U(\Sigma_2) \\ & & \uparrow & & \uparrow \\ \phi|_{\Sigma_1} & : & \Sigma_1 & \rightarrow & \Sigma_2 \end{array}$$

(ii) Φ_j are invertible bundle maps on $U(\Sigma_j)$, $j = 1, 2$.

(iii) D_1 and D_2 , Φ_1 and Φ_2 coincide on the neighborhoods, i.e.,

$$\psi \circ D_1 = D_2 \circ \psi, \quad \psi \circ \Phi_1 = \Phi_2 \circ \psi.$$

We cut M_j along Σ_j and use the map ϕ to glue the pieces together interchanging M_1'' and M_2'' . In this way we obtain the manifolds

$$M_3 := M_1' \cup_{\Sigma} M_2'', \quad M_4 := M_2' \cup_{\Sigma} M_1'',$$

where $\Sigma \cong \Sigma_1 \cong \Sigma_2$. We use the map ψ to cut the bundles E_1, E_2 at Σ_1, Σ_2 and glue the pieces together interchanging $E_1|_{M_1''}$ and $E_2|_{M_2''}$. With this procedure we obtain \mathbb{Z}_2 -graded Hermitian vector bundles $E_3 \rightarrow M_3$ and $E_4 \rightarrow M_4$. At last, we define D_3 and D_4, Φ_3 and Φ_4 by

$$\begin{aligned} D_3 &= \begin{cases} D_1 & \text{on } M_1' \\ D_2 & \text{on } M_2'' \end{cases}, & D_4 &= \begin{cases} D_2 & \text{on } M_2' \\ D_1 & \text{on } M_1'' \end{cases}; \\ \Phi_3 &= \begin{cases} \Phi_1 & \text{on } M_1' \\ \Phi_2 & \text{on } M_2'' \end{cases}, & \Phi_4 &= \begin{cases} \Phi_2 & \text{on } M_2' \\ \Phi_1 & \text{on } M_1'' \end{cases}. \end{aligned}$$

Then by Assumption 7.2.(iii), $D_j + \Phi_j : C_0^\infty(M_j, E_j) \rightarrow C_0^\infty(M_j, E_j)$, $j = 3, 4$ are also \mathbb{Z}_2 -graded essentially self-adjoint Callias-type operators. So again we have two well-defined indexes $\text{ind}(D_3 + \Phi_3)$ and $\text{ind}(D_4 + \Phi_4)$.

7.3. Relative index theorem. As in Subsection 2.17, we define $P_j := D_j + \Phi_j$, $j = 1, 2, 3, 4$. Then we have the following version of the relative index theorem

Theorem 7.4. $\text{ind } P_1 + \text{ind } P_2 = \text{ind } P_3 + \text{ind } P_4$.

The idea of the proof is to use Corollary 6.10 to write $\text{ind } P_j$ as the sum of the indexes on two pieces. However, as one might notice, in our setting, Σ_1 and Σ_2 might not satisfy condition (ii) of Assumption 6.2. So Corollary 6.10 cannot be applied directly. In the next subsection, we construct deformations of the operators P_1 and P_2 which preserve the indexes such that the deformed operators satisfy Assumption 6.2.(ii).

7.5. Deformations of the operators P_1 and P_2 . Let U_j , $j = 1, 2$ denote the neighborhoods $U(\Sigma_j)$ of Σ_j in Subsection 7.1. Since Σ_j are compact hypersurfaces, we can find their relatively compact neighborhoods V_j, W_j satisfying $V_2 = \phi(V_1), W_2 = \phi(W_1)$ and

$$V_j \subset \overline{V_j} \subset W_j \subset \overline{W_j} \subset U_j.$$

Fix smooth functions $f_j : M_j \rightarrow [0, 1]$ such that $f_j \equiv 1$ on $\overline{V_j}$ and $f_j \equiv 0$ outside of W_j . Notice that f_j have compact supports.

For each $t \in [0, \infty)$ define

$$\Phi_{j,t} := (1 + tf_j)\Phi_j,$$

and set

$$P_{j,t} := P_j + tf_j\Phi_j = D_j + (1 + tf_j)\Phi_j = D_j + \Phi_{j,t}.$$

Lemma 7.6. *For $j = 1, 2$, we have*

- (i) *For every $t \geq 0$, the operator $P_{j,t} = D_j + \Phi_{j,t} : C_0^\infty(M_j, E_j^\pm) \rightarrow C_0^\infty(M_j, E_j^\mp)$ is of Callias-type, and, hence, is Fredholm.*
- (ii) *There exists a constant $b > 0$ and a compact subset $K_{j,b} \Subset M_j$, such that $\Sigma_j \subset M_j \setminus K_{j,b}$ and for every $t \geq b$, the essential support of $P_{j,t}$ is contained in $K_{j,b}$.*

Notice, that Lemma 7.6.(ii) implies that for large t , condition (ii) of Assumption 6.2 is satisfied for the operators $P_{j,t}$.

Proof. (i) Direct computation yields

$$\begin{aligned} [D_j, \Phi_{j,t}]_+ &= (1 + tf_j)[D_j, \Phi_j]_+ + \sqrt{-1}t\sigma(D_j)(df_j)\Phi_j, \\ \Phi_{j,t}^2 + [D_j, \Phi_{j,t}]_+ &= (1 + tf_j)^2\Phi_j^2 + (1 + tf_j)[D_j, \Phi_j]_+ + \sqrt{-1}t\sigma(D_j)(df_j)\Phi_j \\ &= \Phi_j^2 + [D_j, \Phi_j]_+ + (t^2f_j^2 + 2tf_j)\Phi_j^2 + tf_j[D_j, \Phi_j]_+ + \sqrt{-1}t\sigma(D_j)(df_j)\Phi_j. \end{aligned}$$

Since both $[D_j, \Phi_j]_+$ and $\sigma(D_j)(df_j)\Phi_j$ are bundle maps, so are $[D_j, \Phi_{j,t}]_+$. Suppose $K_j \Subset M_j$ are the essential supports of P_j . Since the supports of tf_j and df_j both lie in the compact sets $\overline{W_j}$. So $K_j \cup \overline{W_j}$ is still compact and can serve as the essential supports of $P_{j,t}$. Therefore, $P_{j,t}$ are Callias-type operators and, hence, are Fredholm.

(ii) Since K_j are the essential supports of P_j , there exist constants $c_j > 0$, such that

$$|(\Phi_j^2 + [D_j, \Phi_j]_+)(x_j)| \geq c_j, \quad \text{for all } x_j \in M_j \setminus K_j.$$

Since $\overline{V_j}$ are compact sets, $\Phi_j^2 + [D_j, \Phi_j]_+$ have finite lower bounds and Φ_j^2 have positive lower bounds on them. Note that on $\overline{V_j}$, $t^2 f_j^2 \equiv t^2$ and $df_j \equiv 0$. One can find b large enough such that for any $t \geq b$,

$$|(\Phi_{j,t}^2 + [D_j, \Phi_{j,t}]_+)(x_j)| \geq c_j, \quad \text{for all } x_j \in \overline{V_j}.$$

Now we set $K_{j,b} := \overline{K_j \setminus V_j}$. It's easy to see that they are still compact sets and are essential supports of $P_{j,t}$ for $t \geq b$. Clearly, $\Sigma_j \not\subset K_{j,b}$. So we are done. \square

From this lemma, we see that after the deformations of the operators, Σ_j satisfy Assumption 6.2.(ii). It remains to prove the following.

Lemma 7.7. *Let b be the positive constant as in last lemma. Then for $j = 1, 2$,*

$$\text{ind } P_{j,b} = \text{ind } P_j. \quad (7.1)$$

Proof. Using a similar argument as in the proof of Lemma 3.6, for any $t, t' \in [0, \infty)$, $P_{j,t} - P_{j,t'} = (t - t')f_j\Phi_j$ are bounded operators depending continuously on $t - t' \in \mathbb{R}$. By the stability of the index of a Fredholm operator, $\text{ind } P_{j,t}$ are independent of t . Then the lemma follows from setting $t = b$ and $t = 0$. \square

7.8. Proof of Theorem 7.4. Applying the construction of Subsection 7.1 to the operators $P_{1,b}$ and $P_{2,b}$ we obtain operators $P_{3,b}$ and $P_{4,b}$ on M_3 and M_4 respectively. By Lemma 7.7 the indexes of these operators are equal to the indexes of P_3 and P_4 respectively. It follows that it is enough to prove Theorem 7.4 for operators $P_{j,b}$, $j = 1, \dots, 4$. In other words it is enough to prove the theorem for the case when Σ satisfies Assumption 6.2.(ii). Then we can apply Corollary 6.10.

From now on we assume that Σ satisfies Assumption 6.2.(ii) for operators P_1 and P_2 .

As in Section 6, we can define operators $P_{\Sigma_j, j}$, $j = 1, 2$. Let P'_j, P''_j be the restrictions of $P_{\Sigma_j, j}$ to M'_j, M''_j . By Corollary 6.10,

$$\text{ind } P_1 = \text{ind } P'_1 + \text{ind } P''_1,$$

$$\text{ind } P_2 = \text{ind } P'_2 + \text{ind } P''_2.$$

Similarly, we also have

$$\text{ind } P_3 = \text{ind } P'_1 + \text{ind } P''_2,$$

$$\text{ind } P_4 = \text{ind } P'_2 + \text{ind } P''_1.$$

Combining these four equations, we get

$$\text{ind } P_1 + \text{ind } P_2 = \text{ind } P_3 + \text{ind } P_4$$

and complete the proof. \square

REFERENCES

- [1] N. Anghel, *An abstract index theorem on noncompact Riemannian manifolds*, Houston J. Math. **19** (1993), no. 2, 223–237. MR1225459 (94c:58193)
- [2] ———, *On the index of Callias-type operators*, Geom. Funct. Anal. **3** (1993), no. 5, 431–438. MR1233861 (94m:58213)
- [3] R. Bott and R. Seeley, *Some remarks on the paper of Callias: “Axial anomalies and index theorems on open spaces”* [Comm. Math. Phys. **62** (1978), no. 3, 213–234;], Comm. Math. Phys. **62** (1978), no. 3, 235–245.
- [4] M. Braverman, *Index theorem for equivariant Dirac operators on noncompact manifolds*, K-Theory **27** (2002), no. 1, 61–101.
- [5] ———, *New proof of the cobordism invariance of the index*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1095–1101.
- [6] ———, *Cobordism invariance of the index of a transversely elliptic operator*, Appendix J in the book “Moment Maps, Cobordisms, and Hamiltonian Group Actions” by V. L. Ginzburg and V. Guillemin and Y. Karshon. Mathematical Surveys and Monographs, vol. 98, American Mathematical Society, Providence, RI, 2002.
- [7] M. Braverman and L. Cano, *Index theory for non-compact G -manifolds*, Geometric, algebraic and topological methods for quantum field theory, World Sci. Publ., Hackensack, NJ, 2014. pp. 60–94. MR3204959
- [8] J. Bruning and H. Moscovici, *L^2 -index for certain Dirac-Schrödinger operators*, Duke Math. J. **66** (1992), no. 2, 311–336.
- [9] U. Bunke, *A K -theoretic relative index theorem and Callias-type Dirac operators*, Math. Ann. **303** (1995), no. 2, 241–279.
- [10] C. Callias, *Axial anomalies and index theorems on open spaces*, Comm. Math. Phys. **62** (1978), no. 3, 213–235.
- [11] C. Carvalho and V. Nistor, *An Index Formula for Perturbed Dirac Operators on Lie Manifolds*, The Journal of Geometric Analysis **24** (2014), no. 4, 1808–1843.
- [12] V. L. Ginzburg, V. Guillemin, and Y. Karshon, *Cobordism theory and localization formulas for Hamiltonian group actions*, Int. Math. Res. Notices **5** (1996), 221–234.
- [13] M. Gromov and H. B. Lawson Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 295–408.
- [14] V. Guillemin, V. L. Ginzburg, and Y. Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, vol. 98, American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman.
- [15] V. Hardt, A. Konstantinov, and R. Mennicken, *On the spectrum of the product of closed operators*, Math. Nachr. **215** (2000), no. 1, 91–102.
- [16] C. Kottke, *An index theorem of Callias type for pseudodifferential operators*, J. K-Theory **8** (2011), no. 3, 387–417.
- [17] ———, *A Callias-type index theorem with degenerate potentials*, Comm. Partial Differential Equations **40** (2015), no. 2, 219–264.
- [18] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier Analysis, Self-Adjointness*, Academic Press, London, 1975.
- [19] ———, *Methods of modern mathematical physics IV: Analysis of operators*, Academic Press, London, 1978.
- [20] M. Shubin, *Semiclassical asymptotics on covering manifolds and Morse inequalities*, Geom. Funct. Anal. **6** (1996), 370–409.
- [21] R. Wimmer, *An index for confined monopoles*, Comm. Math. Phys. **327** (2014), no. 1, 117–149.

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